

Deformations of surfaces preserving conformal or similarity invariants

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Dedicated to professor Hideki Omori

Introduction

In [11], Burstall, Pedit and Pinkall gave a fundamental theorem of surface theory in Möbius 3-space in modern formulation. Surfaces in Möbius 3-space are determined by conformal Hopf differential and Schwartzian derivative up to conformal transformations. Isothermic surfaces are characterized as surfaces in Möbius 3-space which admit deformations preserving the conformal Hopf differential.

On every surface in Möbius 3-space, a (possibly singular) conformally invariant Riemannian metric is introduced. This metric is called the *Möbius metric* of the surface. The Gaussian curvature of the Möbius metric is called the Möbius curvature. Here we point out that the preservation of Möbius metric is weaker than that of conformal Hopf differential.

Constant mean curvature surfaces (abbreviated as CMC surfaces) in the space forms are typical examples of isothermic surfaces. Bonnet showed that every constant mean curvature surface admits a one-parameter family of isometric deformations preserving the mean curvature. A surface which admits such a family of deformations is called a *Bonnet surface*. Both the isothermic surfaces and Bonnet surfaces are regarded as *geometric* generalizations of constant mean curvature surfaces.

On the other hand, from the viewpoint of integrable system theory, Bobenko introduced the notion of surface with harmonic inverse mean curvature (HIMC surface, in short) in Euclidean 3-space \mathbb{R}^3 . The first named author extended the notion of HIMC surface in \mathbb{R}^3 to that of 3-dimensional space forms [19]. HIMC surfaces have deformation families (associated family) which preserve the conformal structure of the surface and the harmonicity of the reciprocal mean curvature. There exist local bijective conformal correspondences between HIMC surfaces in different space forms.

It should be remarked that while Bonnet surfaces are isothermic, HIMC surfaces are not necessarily isothermic. In fact, the associated family of a Bonnet surface or a HIMC surface preserves the Möbius metric, while

the conformal Hopf differential of a HIMC surface is not preserved in the associated family.

These observations motivate us to study surfaces in Möbius 3-space (or space forms) which admit deformations preserving the Möbius metric. We call such surfaces *Möbius applicable surfaces*.

In this paper we study Möbius applicable surfaces.

First, we shall show the following new characterization of Willmore surfaces (Theorem 1.5):

A surface in Möbius 3-space is Willmore if and only if it is a Möbius applicable surface whose deformation family preserves the Schwarzian derivative.

Next, we shall characterize both Bonnet surfaces and HIMC surfaces in the class of Möbius applicable surfaces in terms of similarity invariants (Theorem 2.4):

A surface in Euclidean 3-space is a Bonnet surface or a HIMC surface if and only if it is a Möbius applicable surface with specific deformation family in which, the ratio of principal curvatures is preserved.

Furthermore we shall give the following characterization of flat Bonnet surfaces (Theorem 2.6):

A Bonnet surface of non-constant mean curvature in Euclidean 3-space is flat if and only if its ratio of principal curvatures or Möbius curvature is constant.

Our characterization results imply that “Bonnet” and “HIMC” are similarity notion. Thus these classes of surfaces fit naturally into similarity geometry.

We emphasize that similarity geometry provide us non-trivial differential geometry of integrable surfaces. In fact, the Burgers hierarchy are derived as deformation of plane curves in similarity geometry.

1 Deformation of surfaces preserving conformal invariants

1.1 Generalities of surface theory in conformal geometry

Let \mathbb{R}^3 be the Euclidean 3-space. The group $\text{Conf}(3)$ of all conformal diffeomorphisms are generated by isometries, dilations and inversions. The conformal compactification \mathcal{M}^3 of \mathbb{R}^3 is called the *Möbius 3-space*. By definition, \mathcal{M}^3 is the 3-sphere equipped with the canonical flat conformal structure.

In this paper, we use the projective lightcone model of the Möbius 3-space introduced by Darboux.

Let \mathbb{R}_1^5 , be the *Minkowski 5-space* with canonical Lorentz scalar product:

$$\langle \xi, \eta \rangle = -\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + \xi_4\eta_4.$$

We denote the natural basis of \mathbb{R}_1^5 by $\{e_0, e_1, \dots, e_4\}$. The unit timelike vector e_0 time-oriens \mathbb{R}_1^5 . The linear isometry group of \mathbb{R}_1^5 is denoted by $O_1(5)$ and called the *Lorentz group* [27]. The *lightcone* \mathcal{L} of \mathbb{R}_1^5 is

$$\mathcal{L} = \{v \in \mathbb{R}_1^5 \mid \langle v, v \rangle = 0, v \neq 0\}.$$

The lightcone has two connected components

$$\mathcal{L}_\pm := \{v \in \mathcal{L} \mid \pm \langle e_0, v \rangle < 0\}.$$

These connected components \mathcal{L}_+ and \mathcal{L}_- are called the *future lightcone* and *past lightcone*, respectively.

For $v \in \mathcal{L}$ and $r \in \mathbb{R}^\times$, clearly, $rv \in \mathcal{L}$. Thus \mathbb{R}^\times acts freely on \mathcal{L} . The quotient $\mathbb{P}(\mathcal{L})$ of \mathcal{L} by the action of \mathbb{R}^\times is called the *projective lightcone*.

The projective lightcone has a conformal structure with respect to which it is conformally equivalent to the unit sphere S^3 with constant curvature 1 metric.

In fact, let us take a unit timelike vector t_0 and set

$$S_{t_0} := \{v \in \mathbb{P}(\mathcal{L}) \mid \langle t_0, v \rangle = -1\}.$$

For $v \in S_{t_0}$, express v as $v = v^\perp + t_0$ so that $v^\perp \perp t_0$. Then

$$0 = \langle v, v \rangle = \langle v^\perp, v^\perp \rangle + \langle t_0, t_0 \rangle = \langle v^\perp, v^\perp \rangle - 1.$$

This implies that the projection $v \mapsto v^\perp$ is an isometry from $S_{t_0} \subset \mathbb{P}(\mathcal{L})$ onto the unit 3-sphere S^3 in the Euclidean 4-space $\mathbb{R}^4 = (\mathbb{R}t_0)^\perp$. This identification induces the following identification:

$$\mathcal{M}^3 \rightarrow \mathbb{P}(\mathcal{L}); v \longmapsto [1 : v]$$

between the Möbius 3-space and the projective lightcone.

More generally, all space forms are realized as conic sections of \mathcal{L} . In fact, for a non-zero vector v_0 , the section S_{v_0} inherits a Riemannian metric of constant curvature $-\langle v_0, v_0 \rangle$.

Definition 1.1 A diffeomorphism of \mathcal{M}^3 is said to be a *Möbius transformation* if it preserves 2-spheres. The Lie group $\text{Möb}(3)$ of Möbius transformations is called the *Möbius group*.

Any conformal diffeomorphism of \mathcal{M}^3 is a Möbius transformation.

The following result is due to Liouville:

Proposition 1.2 *Let $\phi : U \rightarrow V$ be a conformal diffeomorphism between two connected open subsets of \mathcal{M}^3 . Then there exists a unique Möbius transformation Φ of \mathcal{M}^3 such that $\phi = \Phi|_U$.*

The linear action of Lorentz group $O_1(5)$ on \mathbb{R}_1^5 preserves \mathcal{L} and descends to an action on $\mathbb{P}(\mathcal{L})$. Take a unit timelike vector t_0 and $T \in O_1(5)$. Then the restriction of T to S_{t_0} gives an isometry from S_{t_0} onto S_{Tt_0} . This isometry $S_{t_0} \rightarrow S_{Tt_0}$ induces a conformal diffeomorphism on $\mathbb{P}(\mathcal{L})$. These facts together with Liouville's theorem imply that the following sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow O_1(5) \rightarrow \text{Möb}(3) \rightarrow 0$$

is exact. Hence $\text{Möb}(3) \cong O_1^+(5)$, where $O_1^+(5)$ is the subgroup of $O_1(5)$ that preserves \mathcal{L}_\pm . (See [9, Theorem 1.2, 1.3])

The *de Sitter 4-space*

$$S_1^4 = \{v \in \mathbb{R}_1^5 \mid \langle v, v \rangle = 1\}$$

parametrizes the space of all oriented conformal 2-spheres in \mathcal{M}^3 . In fact, take a unit spacelike vector $v \in S_1^4$ and denote by V the 1-dimensional linear subspace spanned by v . Then $\mathbb{P}(\mathcal{L} \cap V^\perp)$ is a conformal 2-sphere in \mathcal{M}^3 . Conversely any conformal 2-sphere can be represented in this form. Via this correspondence, the space of all conformal 2-spheres is identified with S_1^4/\mathbb{Z}_2 . Viewed as a surface $S_{v_0} \cap V^\perp$ of the conic section S_{v_0} , this conformal 2-sphere has the mean curvature vector \mathbb{H}_v

$$\mathbb{H}_v = -v_0^\perp - \langle v_0^\perp, v_0^\perp \rangle v$$

at v , where v_0 is decomposed as $v_0 = v_0^T + v_0^\perp$ according to the orthogonal direct sum $\mathbb{R}_1^5 = V \oplus V^\perp$.

Let $F : M \rightarrow \mathcal{M}^3 = \mathbb{P}(\mathcal{L})$ be a conformal immersion of a Riemann surface into the Möbius 3-space. The *central sphere congruence* (or *mean curvature sphere*) of F is a map $S : M \rightarrow S_1^4$ which assigns to each point $p \in M$, the unique oriented 2-sphere $S(p)$ tangent to F at $F(p)$ which has the same orientation to M and the same mean curvature vector $\mathbb{H}_{S(p)} = \mathbb{H}_p$ at $F(p)$ as F . The pull-back $I_M := \langle dS, dS \rangle$ of the metric of S_1^4 by the central sphere congruence gives a (possibly singular) metric on M and called the *Möbius metric* of (M, F) . The Möbius metric is singular at umbilics. The area functional \mathcal{A}_M of (M, I_M) is called the *Möbius area* of (M, F) . A conformally immersed surface (M, F) is said to be a *Willmore surface* if it is a critical point of the Möbius area functional.

1.2 The integrability condition

Let $F : M \rightarrow \mathcal{M}^3$ be a conformal immersion of a Riemann surface. A *lift* of F is a map $\psi : M \rightarrow \mathcal{L}_+$ into the future lightcone such that $\mathbb{R}\psi(p) = F(p)$ for any $p \in M$. For instance, $\phi := (1, F) : M \rightarrow S_{e_0} \subset \mathcal{L}_+$ is a lift of F . This lift is called the *Euclidean lift* of F . Now let ϕ be the Euclidean lift

of F . Then for any positive function μ on M , $\phi\mu$ is still a lift of F . Direct computation shows that

$$\langle d(\phi\mu), d(\phi\mu) \rangle_1 = \mu^2 \langle dF, dF \rangle_1,$$

where $\langle \cdot, \cdot \rangle_1$ is the constant curvature 1 metric of \mathcal{M}^3 . Take a local complex coordinate z . Then the *normalized lift* ψ with respect to z is defined by the relation:

$$\langle d\psi, d\psi \rangle = dzd\bar{z}.$$

This lift is Möbius invariant. For another local complex coordinate \tilde{z} , the normalized lift $\tilde{\psi}$ with respect to \tilde{z} is computed as $\tilde{\psi} = \psi|\tilde{z}_z|$.

The normalized lift ψ satisfies the following inhomogeneous Hill equation:

$$\psi_{zz} + \frac{c}{2}\psi = \kappa.$$

Under the coordinate change $z \mapsto \tilde{z}$, the coefficients c and κ are changed as

$$\tilde{\kappa} \frac{d\tilde{z}^2}{|d\tilde{z}|} = \kappa \frac{dz^2}{|dz|}.$$

$$\tilde{c}d\tilde{z}^2 = (c - S_z(\tilde{z}))dz^2, \tag{1.1}$$

where $S_z(\tilde{z})$ is the *Schwarzian derivative* of \tilde{z} with respect to z . Here we recall that the Schwarzian derivative $S_z(f)$ of a meromorphic function f on M is defined by

$$S_z(f) := \left(\frac{f_{zz}}{f_z} \right)_z - \frac{1}{2} \left(\frac{f_{zz}}{f_z} \right)^2.$$

Moreover two meromorphic functions f and g are *Möbius equivalent*, i.e., related by a linear fractional transformation:

$$g = \frac{af + b}{cf + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\mathbb{C}$$

if and only if their Schwartzian derivatives $S_z(f) = S_z(g)$ agree.

Now we denote by L the 1-density bundle of M :

$$L := (K \otimes_{\mathbb{C}} \overline{K})^{-\frac{1}{2}}, \quad K \text{ is the canonical bundle of } M.$$

The transformation law (1.1) implies that $\kappa dz^2/|dz|$ is a section of LK^2 , i.e., an L -valued quadratic differential on M . This section is called the *conformal Hopf differential* of (M, F) . The differential cdz^2 is called the *Schwarzian* of (M, F) . The coefficient function c is also called the Schwarzian.

Note that the conformal Hopf differential vanishes identically if and only if M is totally umbilical.

The integrability condition for a conformal immersion $F : M \rightarrow \mathcal{M}^3$ is given in terms of κ and c as follows:

$$\begin{cases} \frac{1}{2}c_{\bar{z}} = 3\bar{\kappa}_z\kappa + \bar{\kappa}\kappa_z, \\ \text{Im}(\kappa_{\bar{z}\bar{z}} + \frac{1}{2}\bar{c}\kappa) = 0. \end{cases} \quad (1.2)$$

These equations are called, the *conformal Gauss equation* and the *conformal Codazzi equation*, respectively.

The Möbius metric $I_{\mathcal{M}}$ is represented by

$$I_{\mathcal{M}} = 4|\kappa|^2 dzd\bar{z}. \quad (1.3)$$

The Euler-Lagrange equation for the Möbius area functional $\mathcal{A}_{\mathcal{M}}$ is called the *Willmore surface equation* and given in terms of κ and c as follows if [11, p. 51]:

$$\kappa_{\bar{z}\bar{z}} + \frac{1}{2}\bar{c}\kappa = 0. \quad (1.4)$$

1.3 Deformation of surfaces preserving the Schwarzian derivative or the conformal Hopf differential

Generally speaking, the conformal Hopf differential alone determines surfaces in \mathcal{M}^3 . However, there are the only exceptional surfaces—*isothermic surfaces* [10]. Isothermic surfaces are defined as surfaces in \mathcal{M}^3 conformally parametrized by their curvature lines away from umbilics. Away from umbilics, there are holomorphic coordinates in which the conformal Hopf differential is real valued. Such holomorphic coordinates (and their associated real coordinates) are called *isothermic coordinates*.

Now let (M, F) be an isothermic surface parametrized by an isothermic coordinate z . Then under the deformation:

$$c \longrightarrow c_r := c + r, \quad r \in \mathbb{R},$$

the conformal Gauss-Codazzi equations

$$c_{\bar{z}} = 4(\kappa^2)_z, \quad \text{Im}(\kappa_{\bar{z}\bar{z}} + \frac{1}{2}\bar{c}\kappa) = 0$$

are invariant. Hence, as in the case of CMC surfaces, one obtains a 1-parameter family $\{F_r\}$ of deformations through $F = F_0$ preserving the conformal Hopf differential κ . Since all c_r are distinct, the surfaces $\{F_r\}$ are non-congruent, each other. The family $\{F_r\}$ is referred as the *associated family* of an isothermic surface (M, F) . The correspondence $F \longrightarrow F_r$ is called the *T-transformation* by Bianchi [3]. The *T-transformation* was independently introduced by Calapso [12] and is also called the *Calapso transformation*.

The existence of deformations preserving the conformal Hopf differential characterizes isothermic surfaces as follows:

Theorem 1.3 ([11]) *A surface in \mathcal{M}^3 is isothermic if and only if it has deformations preserving the conformal Hopf differential.*

Corollary 1.4 ([11]) *Let $F_1, F_2 : M \rightarrow \mathcal{M}^3$ be two non-congruent surfaces with same conformal Hopf differential. Then both F_1 and F_2 belong to the same associated family of an isothermic surface.*

On the other hand, for deformations preserving Möbius metric and Schwarzian, we have the following *new* characterization of Willmore surfaces.

Theorem 1.5 *A surface in \mathcal{M}^3 is Willmore if and only if it has Möbius-isometric deformations preserving the Schwarzian derivative.*

Proof. Let F be a surface in \mathcal{M}^3 with the Schwarzian derivative c and the conformal Hopf differential κ . If F has deformation preserving the Möbius metric $I_{\mathcal{M}}$ and c , there exists an S^1 -valued function λ such that $\lambda\kappa$ and c satisfy the conformal Gauss equation. Combining this with the conformal Gauss equation for F , we have

$$3\bar{\lambda}_z\lambda + \bar{\lambda}\lambda_z = 0, \quad (1.5)$$

which implies that $\lambda^3\bar{\lambda}$ is holomorphic and hence λ is an S^1 -valued constant. Since $\lambda\kappa$ and c satisfy the conformal Codazzi equation, combining this with the conformal Codazzi equation for F , we have

$$\kappa_{z\bar{z}} + \frac{1}{2}\bar{c}\kappa = 0, \quad (1.6)$$

which implies that F is Willmore. \square

Remark 1.6 É. Cartan formulated a general theory of deformation of submanifolds in homogeneous spaces. The classical deformation problems (also called *applicability* of submanifolds in classical literatures) in Euclidean, projective and conformal geometry are covered by Cartan's framework [13]-[14].

According to Griffiths [21] and Jensen [24], two immersions $F_1, F_2 : M \rightarrow G/K$ of a manifold into a homogeneous manifold are said to be *k-th order deformation* of each other if there exists a smooth map $g : M \rightarrow G$ such that, for every $p \in M$, the Taylor expansions about p of F_2 and $g(p)F_1$ agree through k -th order terms. An immersion $F : M \rightarrow G/K$ is said to be *deformable of order k* if it admits a non-trivial k -th order deformation.

Musso [25] showed that a conformal immersion of a Riemann surface M into the Möbius 3-space is 2nd order deformable if and only if it is isothermic.

Remark 1.7 (Special isothermic surfaces) Among isothermic surfaces in \mathbb{R}^3 , Darboux [18] distinguished the class of *special isothermic surfaces*. An isothermic surface $F : M \rightarrow \mathbb{R}^3$ with first and second fundamental forms;

$$\text{I} = e^\omega(dx^2 + dy^2), \quad \text{II} = e^\omega(k_1dx^2 + k_2dy^2)$$

is called *special* of type (A, B, C, D) if its mean curvature H satisfies the equation:

$$4e^\omega |\nabla H|^2 + m^2 + 2Am + 2BH + 2C\ell + D = 0,$$

where $\ell = 2e^\omega \sqrt{H^2 - K}$, $m = -H\ell$ and A, B, C, D are real constants. Constant mean curvature surfaces are particular examples of special isothermic surface. Special isothermic surfaces with $B = 0$ are conformally invariant. Moreover, Bianchi [2] and Calapso [12] showed that an umbilic free isothermic surface in \mathcal{M}^3 is special with $B = 0$ if and only if it is conformally equivalent to a constant mean curvature surface in space forms. For modern treatment of special isothermic surfaces and their Darboux transformations, we refer to [26]. In [1], Bernstein constructed non-special, non-canal isothermic tori in \mathcal{M}^3 with spherical curvature lines.

Let $F : M \rightarrow \mathcal{M}^3$ be a conformal immersion. Then F is said to be a *constrained Willmore surface* if it is a critical point of the Möbius area functional under (compactly supported) conformal variations.

Proposition 1.8 ([11]) *$F : M \rightarrow \mathcal{M}^3$ is constrained Willmore if and only if there exists a holomorphic quadratic differential qdz^2 such that*

$$\kappa_{\bar{z}\bar{z}} + \frac{1}{2}\bar{c}\kappa = \operatorname{Re}(\bar{q}\kappa). \quad (1.7)$$

The constrained Willmore surface equation (1.7) has the following deformation:

$$\kappa \longrightarrow \kappa_\lambda := \lambda\kappa, \quad c \longrightarrow c_\lambda := c + (\lambda^2 - 1)q, \quad q \longrightarrow q_\lambda := \lambda q,$$

for $\lambda \in S^1$.

Hence we obtain a one-parametric conformal deformation family $\{F_\lambda\}$ of a constrained Willmore surface (M, F) . This family is referred as the *associated family* of F .

Obviously, for Willmore surfaces ($q = 0$), the associated family preserves the Schwarzian.

The following characterization of constrained Willmore surfaces can be verified in a way similar to the proof of Theorem 1.5:

Proposition 1.9 *A surface $F : M \rightarrow \mathcal{M}^3$ has a deformation of the form*

$$\kappa \longrightarrow \lambda\kappa, \quad c \longrightarrow c + r$$

for some S^1 -valued function λ and a holomorphic quadratic differential rdz^2 if and only if M is a constrained Willmore surface.

Remark 1.10 A classical result by Thomsen says that a surface is isothermic Willmore if and only if it is minimal in a space form ([8], [23, Theorem

3.6.7], [29]). Constant mean curvature surfaces in space forms are isothermic and constrained Willmore. Richter [28] showed that in the case of immersed tori in \mathcal{M}^3 , every isothermic constrained Willmore tori are constant mean curvature tori in some space forms. In contrast to Thomsen's result, the assumption "tori" is essential for Richter's result. In fact, Burstall constructed isothermic constrained Willmore cylinders which are not realized as constant mean curvature surfaces in any space forms. See [6].

2 Deformation of surfaces preserving similarity invariants

As we saw in the preceding section, preservation of conformal Hopf differential is a strong restriction in the study of deformation of surfaces. Clearly, the preservation of Möbius metric is weaker than that of conformal Hopf differential. In this section we study deformation of surfaces preserving the Möbius metric.

2.1 Möbius invariants via metrical language

First, we discuss relations between metrical invariants and Möbius invariants.

Let $F : M \rightarrow \mathbb{R}^3$ be a conformal immersion of a Riemann surface into the Euclidean 3-space. Denote by I the first fundamental form (induced metric) of M . The Levi-Civita connections D of \mathbb{R}^3 and ∇ of M are related by the *Gauss equation*:

$$D_X F_* Y = F_*(\nabla_X Y) + \mathbb{I}(X, Y)\mathbf{n}.$$

Here \mathbf{n} is the unit normal vector field. The symmetric tensor field \mathbb{I} is the *second fundamental form* derived from \mathbf{n} .

The trace free part of the second fundamental form is given by $\mathbb{I} - HI$, where H is the mean curvature function. Define a function h by $h := \sqrt{H^2 - K}$. This function h is called the *Calapso potential*.

Then one can check that the normal vector field \mathbf{n}/h and the symmetric tensor field $h^2 I$ are invariant under the conformal change of the ambient Euclidean metric. Moreover the trace free symmetric tensor field

$$\mathbb{I}_{\mathcal{M}} := h(\mathbb{I} - HI)$$

is also conformally invariant. It is easy to see that $h^2 I$ coincides with the *Möbius metric* $I_{\mathcal{M}}$ of (M, F) . The pair $(I_{\mathcal{M}}, \mathbb{I}_{\mathcal{M}})$ is called the *Fubini's conformally invariant fundamental forms*. The Gaussian curvature $K_{\mathcal{M}}$ of $(M, I_{\mathcal{M}})$ is called the *Möbius curvature* of (M, F) . The Möbius area functional $\mathcal{A}_{\mathcal{M}}$ of $(M, I_{\mathcal{M}})$ is computed as

$$\mathcal{A}_{\mathcal{M}} = \int_M (H^2 - K) dA_I.$$

Now let us take a local complex coordinate z and express the first fundamental form as $I = e^\omega dz d\bar{z}$. The (metrical) *Hopf differential* is defined by

$$Q^\# := Q dz^2, \quad Q = \langle F_{zz}, \mathbf{n} \rangle.$$

Then the conformal Hopf differential and the metrical one are related by the formula:

$$\kappa = Q e^{-\frac{\omega}{2}}. \quad (2.1)$$

The Schwarzian derivative is represented as

$$c = \omega_{zz} - \frac{1}{2}(\omega_z)^2 + 2HQ.$$

2.2 Similarity geometry

The similarity geometry is a subgeometry of Möbius geometry whose symmetry group is the *similarity transformation group*:

$$\text{Sim}(3) = \text{CO}(3) \ltimes \mathbb{R}^3,$$

where $\text{CO}(3)$ is the linear conformal group

$$\text{CO}(3) = \{A \in \text{GL}_3\mathbb{R} \mid \exists c \in \mathbb{R}; {}^tAA = cE\}.$$

Let $F : M \rightarrow \mathbb{R}^3$ be an immersed surface with unit normal \mathbf{n} as before.

Under the similarity transformation of \mathbb{R}^3 , Levi-Civita connections D and ∇ are invariant. Hence the vector valued second fundamental form $\mathbb{I}\mathbf{n}$ is similarity invariant. The shape operator $S = -d\mathbf{n}$ itself is not similarity invariant, but the ratio of principal curvatures are invariant. It is easy to see that the constancy of the ratio of principal curvatures is equivalent to the constancy of K/H^2 . The function K/H^2 is similarity invariant. The principal directions are yet another similarity invariant.

2.3 Deformation of surfaces preserving the Möbius metric and the ratio of principal curvatures

Let $F : M \rightarrow \mathbb{R}^3$ be a surface in Euclidean 3-space. Then the Gauss-Codazzi equations of (M, F) are given by

$$\begin{cases} \omega_{z\bar{z}} + \frac{1}{2}H^2e^\omega - 2|Q|^2e^{-\omega} = 0, \\ Q_{\bar{z}} = \frac{1}{2}H_z e^\omega. \end{cases} \quad (2.2)$$

The Gauss-Codazzi equations imply the following fundamental fact due to Bonnet.

Proposition 2.1 ([7]) *Every non-totally umbilical constant mean curvature surface admits a one-parameter isometric deformation preserving the mean curvature.*

Here we exhibit two examples of surfaces which admit deformations preserving the Möbius metric.

Example 2.2 (Bonnet surfaces) Let $F : M \rightarrow \mathbb{R}^3$ be a Bonnet surface. Namely (M, F) admits a non-trivial isometric deformation $F \rightarrow F_\lambda$ preserving the mean curvature. The deformation family $\{F_\lambda\}$ is called the *associated family* of (M, F) .

Since all the members F_λ have the same metric and the mean curvature, they have the same Möbius metric. Note that the conformal Hopf differential is not preserved under the deformation.

Example 2.3 (HIMC surfaces) A surface $F : M \rightarrow \mathbb{R}^3$ is said to be a *surface with harmonic inverse mean curvature* (HIMC surface, in short) if its inverse mean curvature function $1/H$ is a harmonic function on M [4]. Since $1/H$ is harmonic, H can be expressed as $1/H = h + \bar{h}$ for some holomorphic function h . The associated family $\{F_\lambda\}$ of a HIMC surface F is given by the following metrical data $(I_\lambda, H_\lambda, Q_\lambda)$:

$$I_\lambda = e^{\omega_\lambda} dz d\bar{z}, \quad e^{\omega_\lambda} = \frac{e^\omega}{(1 - 2\sqrt{-1}\bar{h}t)^2(1 + 2\sqrt{-1}ht)^2},$$

$$\frac{1}{H_\lambda} = h_\lambda + \bar{h}_\lambda, \quad h_\lambda = \frac{h}{1 + 2\sqrt{-1}ht},$$

$$Q_\lambda = \frac{Q}{(1 + 2\sqrt{-1}ht)^2}, \quad \lambda = \frac{1 - 2\sqrt{-1}\bar{h}t}{1 + 2\sqrt{-1}ht}, \quad t \in \mathbb{R}.$$

From these, we have

$$(H_\lambda^2 - K_\lambda) = (1 - 2\sqrt{-1}\bar{h}t)(1 + 2\sqrt{-1}ht)(H^2 - K),$$

Hence

$$(H_\lambda^2 - K_\lambda)e^{\omega_\lambda} = (H^2 - K)e^\omega.$$

Thus the Möbius metric is preserved under the deformation $F \rightarrow F_\lambda$. On the other hand, the conformal Hopf differential is not preserved under the deformation. In fact, the conformal Hopf differential of F_λ is

$$\kappa_\lambda := Q_\lambda e^{-\frac{\omega_\lambda}{2}} = \kappa \frac{1 - 2\sqrt{-1}\bar{h}t}{1 + 2\sqrt{-1}ht}.$$

Clearly κ_λ is not preserved under the deformation.

While Bonnet surfaces are isothermic, HIMC surfaces are not necessarily so. The dual surfaces of Bonnet surfaces are isothermic HIMC surfaces. Since the associated families of Bonnet surface or isothermic HIMC surfaces do not preserve the conformal Hopf differential, these families differ from the T -transformation families. Note that T -transformations are only well defined up to Möbius transformations [9, section 2.2.3].

Now we prove the following theorem which characterizes Bonnet surfaces and HIMC surfaces in the class of surfaces which posses Möbius metric preserving deformations. We call such surfaces *Möbius applicable surfaces*.

Theorem 2.4 *Let F be a surface in \mathbb{R}^3 which has deformation preserving the Möbius metric and the ratio of principal curvatures. Then the deformation is given by*

$$e^\omega \rightarrow |\lambda|^2 e^\omega, \quad H \rightarrow \frac{1}{|\lambda|} H, \quad Q \rightarrow \lambda Q, \quad (2.3)$$

where λ is a function with $|\lambda| = |f|$ for some holomorphic function f . Moreover if $|\lambda| = 1$ (respectively λ is holomorphic), then F is a Bonnet surface (respectively a HIMC surface).

Proof. Note that the quantities $|Q|^2 e^{-\omega}$ and $e^{-\omega}/H^2$ are invariant under the deformation, which implies that the deformation is given by as above for some function λ (see (2.1)). From the Gauss equation we have

$$(\log |\lambda|^2)_{z\bar{z}} = 0,$$

which implies that $|\lambda| = |f|$ for some holomorphic function f .

If $|\lambda| = 1$, the deformation is nothing but the isometric deformation preserving the mean curvature. Hence F is a Bonnet surface.

If λ is holomorphic, putting $(H')^2 = H^2/|\lambda|^2$ and differentiating it by z , we have

$$2H'H'_z = -\frac{\bar{\lambda}\lambda_z}{|\lambda|^4} H^2 + \frac{2}{|\lambda|^2} H H_z.$$

Note that $Q \neq 0$ since F is umbilic-free. Combining the Codazzi equations for F and the surface obtained by deformation, we have

$$H'_z = \frac{1}{\lambda} H_z. \quad (2.4)$$

Hence we have

$$H' = -\frac{\lambda_z}{2\lambda^2} \frac{H^2}{H_z} + \frac{1}{\lambda} H.$$

Differentiating it by \bar{z} and using (2.4) again, we have

$$H_{z\bar{z}} - \frac{2|H_z|^2}{H} = 0,$$

which implies that F is a HIMC surface. \square

2.4 Flat Bonnet surfaces

Let M be a Bonnet surface in \mathbb{R}^3 . Then away from umbilics, there exists an isothermic coordinate z such that the Gauss-Codazzi equations of M reduces to the following third order ordinary differential equation (*Hazzidakis equation* [22]):

$$\left\{ \left(\frac{H_{ss}}{H_s} \right)_s - H_s \right\} R^2 = 2 - \frac{H^2}{H_s}, \quad H_s < 0, \quad (2.5)$$

where $s = z + \bar{z}$ and the coefficient function $R(s)$ is one of the following function [5, p. 30]:

$$R_A(s) = \frac{\sin(2s)}{2}, \quad R_B(s) = \frac{\sinh(2s)}{2}, \quad R_C(s) = s.$$

The modulus $|Q|$ of the metrical Hopf differential Qdz^2 is given by

$$|Q(z, \bar{z})| = \frac{1}{R(s)^2}. \quad (2.6)$$

A Bonnet surface is said to be of *type A*, *B* or *C*, respectively if away from critical points of the mean curvature, it is determined by a solution to Hazzidakis equation with coefficient R_A , R_B or R_C ([5, Definition 3.2.1], [15]).

Proposition 2.5 ([5], [20]) *Flat Bonnet surfaces in \mathbb{R}^3 are of C-type.*

Flat Bonnet surfaces are characterized as follows in terms of conformal (Möbius) or similarity invariants.

Theorem 2.6 *A Bonnet surface in \mathbb{R}^3 with non-constant mean curvature is flat if and only if the Möbius curvature or the ratio of the principal curvatures is constant.*

Proof. First we consider Bonnet surfaces with constant ratio of principal curvatures. By the assumption the function K/H^2 is constant. Computing K/H^2 by using (2.5) and (2.6), one can deduce that $K = 0$ if K/H^2 is constant.

Next, the Möbius curvature $K_{\mathcal{M}}$ is computed as

$$K_{\mathcal{M}} = \frac{1}{H_s} (\log H_s)_{ss}$$

by using the Hazzidakis equation (2.5).

If $K_{\mathcal{M}}$ is constant, a direct computation shows that the solution of (2.5) is

$$H = -\frac{2}{K_{\mathcal{M}}} \frac{1}{s}$$

with $K_{\mathcal{M}} < 0$. Hence the surface is flat. \square

Appendices

A.1. Curves in similarity geometry.

Let us consider plane curve geometry in the 2-dimensional similarity geometry $(\mathbb{R}^2, \text{Sim}(2))$. Here $\text{Sim}(2)$ denotes the similarity transformation group of \mathbb{R}^2 .

Let $\gamma(s)$ be a regular curve on \mathbb{R}^2 parametrized by the Euclidean arclength σ . Then the $\text{Sim}(2)$ -invariant parameter s is the *angle function* $s = \int^\sigma \kappa_E(\sigma) d\sigma$, where κ_E is the Euclidean curvature function. The $\text{Sim}(2)$ -invariant curvature κ_S is given by $\kappa_S = (\kappa_E)_\sigma / \kappa_E^2$. Obviously, every circle is a curve of similarity curvature 0. The $\text{Sim}(2)$ -invariant frame field $\mathcal{F} = (T, N)$ is given by

$$T = \gamma_s, \quad N = T_s + \kappa_S T.$$

The Frenet-Serret equation of \mathcal{F} is

$$\mathcal{F}^{-1} \frac{d\mathcal{F}}{ds} = \begin{pmatrix} -\kappa_S & -1 \\ 1 & -\kappa_S \end{pmatrix}.$$

Now let us consider plane curves of nonzero constant similarity curvature.

Put $\kappa_S = c_1$ (constant). Then we have $1/\kappa_E = (-c_1)\sigma + c_2$, namely γ is a curve whose inverse Euclidean curvature $1/\kappa_E$ is a linear function of the Euclidean arclength parameter. Thus γ is a log-spiral (if $c_1 \neq 0$) or a circle ($c_1 = 0, c_2 \neq 0$).

These curves provide fundamental examples of Bonnet surfaces as well as HIMC surfaces. In fact, let γ be a plane curve of constant similarity curvature. Then cylinder over γ is a flat Bonnet surface in \mathbb{R}^3 as well as a flat HIMC surface in \mathbb{R}^3 . Generally, the Hazzidakis equation of Bonnet or isothermic HIMC surfaces reduces to Painlevé equations of type III, V or VI. The solutions to log-spiral cylinder are elementary function solutions to these Painlevé equations. (see [5], [20]).

A.2. Time evolutions

Let us consider the time evolution of a plane curve $\gamma(s)$ in similarity geometry.

Denote by $\gamma(s; t)$ the time evolution which preserves the similarity arclength parameter s ;

$$\frac{\partial}{\partial t} \gamma(s; t) = gN + fT$$

Then the similarity curvature $u = \kappa_S$ obeys the following partial differential equation:

$$u_t = f_{sss} - 2uf_{ss} - (3u_s - u^2 - 1)f_s - (u_{ss} - 2uu_s)f + au_s, \quad a \in \mathbb{R}.$$

In particular, if we choose $f = -1$, $a = 0$, then the time evolution of u obeys the *Burgers equation*:

$$u_t = u_{ss} - 2uu_s.$$

More generally, the Burgers hierarchy is induced by the above time evolution, see [16, pp. 17–18]. Space curves in similarity geometry and their time evolution, we refer to [17].

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